

How Hard Is It to Control a Group?

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Abstract

Group identification models situations where a set of individuals are asked to determine who among themselves are socially qualified. In this paper, we study the complexity of three group control problems, namely GROUP CONTROL BY ADDING INDIVIDUALS, GROUP CONTROL BY DELETING INDIVIDUALS and GROUP CONTROL BY PARTITION OF INDIVIDUALS for various social rules. In these problems, an external agent has an incentive to make a given subset of individuals socially qualified by adding, deleting or partition of individuals. We achieve both polynomial-time solvability results and NP-hardness results. In addition, we study the NP-hard problems from the parameterized complexity perspective, and obtain several fixed-parameter tractability results. On the other hand, we prove that some social rules are immune to these problems.

1 Introduction

Decision making plays an important role in multi-agent systems. For instance, a set of agents (or robots) need to complete a task cooperatively. Due to some reasons (e.g., in order to minimize the cost of the resources), only a few agents can take the job. In this case, all agents need to make a joint decision of which agents are going to take the job. In this paper, we study such a decision making model, in which a set N of individuals desire to select a subset of individuals of N . In particular, each individual qualifies or disqualifies every individual in N , and then a social rule is applied to select the socially qualified individuals. This model has been widely studied under the name *group identification* in economics [5, 6, 21, 23]. In particular, the liberal rule, the consent rules, the consensus-start-respecting rule (CSR) and the liberal-start-respecting rule (LSR) have been extensively studied in the literature [5, 20, 23]. Due to the liberal rule, an individual is socially qualified if and only if this individual qualifies herself. Consent rules are a class of social rules, where each of them is characterized by two positive integers s and t . Moreover, if an individual qualifies herself, then this individual is socially qualified if and only if there are at least $s - 1$ other individuals who also qualify her. On the other hand, if the individual disqualifies herself, then this individual is not socially qualified if and only if there are at least $t - 1$ other individuals who also disqualify her. The CSR and the LSR social rules recursively determine the socially qualified individuals. In the beginning, the set K^L of individuals each of whom qualifies herself are considered LSR socially qualified, while the set K^C of individuals each of whom is qualified by all individuals are considered CSR socially qualified. Then, in each iteration for the social rule LSR (resp. CSR), an individual a is added to K^L (resp. K^C) if there is an individual in K^L (resp. K^C) qualifying a . The iteration terminates until no new individual can be added to K^L (resp. K^C), and the socially qualified individuals are the ones in K^L (resp. K^C).

In this paper, we consider the problems where an external (strategic) agent has an incentive to control the results by adding, deleting or partition of individuals. In particular, in each problem the external agent has a set S of objective individuals whom he wants to make socially qualified. Though it is possible for the external agent to change the result in many cases, the external agent might give up controlling the group identification procedure if he realizes that it would take quite a long time to find out how to change the result successfully. Motivated by this argument, we study the complexity of these problems for the liberal rule, the consent rules, the CSR and the LSR rules. We achieve both polynomial-time solvability results and NP-hardness results for these problems. In particular, we obtain dichotomy results for all problems considered in this paper for consent rules, with respect to the values of s and t . In addition, we study the NP-hard problems from the parameterized complexity point of view, and obtain several fixed-parameter tractability results, with respect to the number of objective individuals. See Table 1 for a summary of our main results.

1.1 Preliminaries

Social Rule. Let N be a set of individuals. A *profile* $\varphi : N \times N \rightarrow \{0, 1\}$ over N is a mapping such that $\varphi(a, a') = 1$ means that individual $a \in N$ *qualifies* individual $a' \in N$. A *social rule* is a function f which associates a pair (φ, T) of each profile φ over N and a subset $T \subseteq N$ of individuals with a subset $f(\varphi, T) \subseteq T$. We call the individuals in $f(\varphi, T)$ the *socially qualified individuals* of T with respect to f and φ . In this paper, we mainly study the following social rules.

Liberal rule f^L . An individual is socially qualified if and only if this individual qualifies herself. That is, for every $T \subseteq N$ and every individual $a \in T$, $a \in f^L(\varphi, T)$ if and only if $\varphi(a, a) = 1$.

Consent rules $f^{(s,t)}$. Each consent rule $f^{(s,t)}$ is specified by two positive integers s and t such that for every $T \subseteq N$ and every individual $a \in T$,

- (1) if $\varphi(a, a) = 1$, then $a \in f^{(s,t)}(\varphi, T)$ if and only if $|\{a' \in T \mid \varphi(a', a) = 1\}| \geq s$, and
- (2) if $\varphi(a, a) = 0$, then $a \notin f^{(s,t)}(\varphi, T)$ if and only if $|\{a' \in T \mid \varphi(a', a) = 0\}| \geq t$.

The two positive integers s and t are referred to as the *consent quotas* of the consent rule $f^{(s,t)}$. It is easy to see that the consent rule $f^{(1,1)}$ is exactly the liberal rule [23].

We remark that in the original definition of consent rules by Samet and Schmeidler [23], there is an additional condition $s + t \leq n + 2$ for consent quotas s and t to satisfy, where n is the number of individuals. Indeed, Samet and Schmeidler studied the consent rules for a fixed set of individuals, and the condition $s + t \leq n + 2$ is crucial for the consent rules to satisfy the *monotonicity property*. Roughly, a social rule is *monotonic* if a socially qualified individual a is still socially qualified when someone who disqualifies a turns to qualify a . We refer to [23] for further details. Since our paper is mainly concerned with complexity of strategic group control problems, we drop this condition from the definition of the consent rules (we indeed achieve results for a more general class of social rules that encapsules the original consent rules defined in [23]). When studying the group control problems for the consent rules $f^{(s,t)}$ we assume that the consent quotas s and t remain the same, that is, they do not change after adding new individuals, deleting old ones, or partitioning of individuals.

Consensus-start-respecting rule f^{CSR} . For every $T \subseteq N$, this rule determines the socially qualified individuals iteratively. First, every individual who is qualified by all individuals is considered socially qualified. Then, in each iteration, all individuals which are qualified by at least one of the currently socially qualified individuals are added to the set of socially qualified individuals. The iteration terminates until no new individual is added. Precisely, for every $T \subseteq N$ let

$$K_0^C(\varphi, T) = \{a \in T \mid \forall a' \in T, \varphi(a', a) = 1\}.$$

For each positive integer $\ell = 1, 2, \dots$, let

$$K_\ell^C(\varphi, T) = K_{\ell-1}^C(\varphi, T) \cup \{a \in T \mid \exists a' \in K_{\ell-1}^C(\varphi, T), \varphi(a', a) = 1\}.$$

Then $f^{CSR}(\varphi, T) = K_\ell^C(\varphi, T)$ for some ℓ such that $K_\ell^C(\varphi, T) = K_{\ell-1}^C(\varphi, T)$.

Liberal-start-respecting rule f^{LSR} . This rule is similar to f^{CSR} with the only difference that the initial socially qualified individuals are those who qualify themselves. In particular, for every $T \subseteq N$, let

$$K_0^L(\varphi, T) = \{a \in T \mid \varphi(a, a) = 1\}.$$

For each positive integer $\ell = 1, 2, \dots$, let

$$K_\ell^L(\varphi, T) = K_{\ell-1}^L(\varphi, T) \cup \{a \in T \mid \exists a' \in K_{\ell-1}^L(\varphi, T), \varphi(a', a) = 1\}.$$

Then $f^{LSR}(\varphi, T) = K_\ell^L(\varphi, T)$ for some ℓ such that $K_\ell^L(\varphi, T) = K_{\ell-1}^L(\varphi, T)$.

Problem Definition. In this paper, we mainly study the complexity of the following problems.

GROUP CONTROL BY ADDING INDIVIDUALS (GCAI)

Input: A 6-tuple (f, N, φ, S, T, k) of a social rule f , a set N of individuals, a profile φ over N , two nonempty subsets $S, T \subseteq N$ such that $S \subseteq T$ and $S \not\subseteq f(\varphi, T)$, and a positive integer k .

Question: Is there a subset $U \subseteq N \setminus T$ such that $|U| \leq k$ and $S \subseteq f(\varphi, V)$ with $V = T \cup U$?

GROUP CONTROL BY DELETING INDIVIDUALS (GCDI)

Input: A 5-tuple (f, N, φ, S, k) of a social rule f , a set N of individuals, a profile φ over N , a nonempty subset $S \subseteq N$ such that $S \not\subseteq f(\varphi, N)$, and a positive integer k .

Question: Is there a subset $U \subseteq N \setminus S$ such that $|U| \leq k$ and $S \subseteq f(\varphi, V)$ with $V = N \setminus U$?

GROUP CONTROL BY PARTITION OF INDIVIDUALS (GCPI)

Input: A 4-tuple (f, N, φ, S) of a social rule f , a set N of individuals, a profile φ over N , and a nonempty subset $S \subseteq N$ such that $S \not\subseteq f(\varphi, N)$.

Question: Is there a subset $U \subseteq N$ such that $S \subseteq f(\varphi, V)$ with $V = f(\varphi, U) \cup f(\varphi, N \setminus U)$?

A social rule is *immune* to a problem defined as above if it is impossible to make a non-socially qualified individual $a \in S$ socially qualified by carrying out the operations (adding/deleting/partition of individuals) in the problems, that is, there are only NO-instance. If a social rule is not immune to a problem defined above, we say it is *susceptible* to the problem.

Graph. An *undirected graph* is a tuple (W, E) where W is the *vertex set* and E is the *edge set*. A vertex v *dominates* a vertex u if there is an edge between v and u . A vertex subset A *dominates* another vertex subset B , if for every vertex $u \in B$ there is a vertex $v \in A$ that dominates u . A *directed graph* is a tuple (W, A) where W is the vertex set and A is the *arc set*. An

	consent rules $f^{(s,t)}$						CSR	LSR
	$s = 1$			$s \geq 2$				
	$t = 1$	$t = 2$	$t \geq 3$	$t = 1$	$t = 2$	$t \geq 3$		
GCAI	I	I	I	NP (+)	NP (+)	NP(+)	NP	NP
GCDI	I	P	NP (+)	I	P	NP (+)	I	I
GCPI	I	NP	NP	I	NP	NP	I	I

Table 1: A summary of our results. In the table, “NP” means “NP-hard”, “P” means “polynomial-time solvable”, and “I” means “immune”. The consent rule $f^{(1,1)}$ is exactly the liberal rule. The immunity results for the consent rules are from Theorems 1 and 2. The polynomial-time solvability results for the consent rules are from Theorem 3. The NP-hardness results of GCAI and GCDI for consent rules are from Theorem 4. The NP-hardness results of GCPI for consent rules are from Theorem 5. The NP-hardness results for the CSR and LSR rules are from Theorem 6. The immunity results for the CSR and LSR rules are from Theorem 7. The NP-hardness results with a “+” next to them mean that the problems are FPT with respect to $|S|$. All FPT results are from Theorem 8.

independent set I of a digraph (resp. an undirected graph) is a vertex subset such that there is no arc (resp. edge) between every two vertices in I . A *directed* (resp. *an undirected*) *bipartite graph* is a digraph (resp. an undirected graph) whose vertex set can be partitioned into two independent sets. We denote by $(L \uplus R, A)$ (resp. $(L \uplus R, E)$) a directed (resp. an undirected) bipartite graph with (L, R) being a partition of its vertex set such that both L and R are independent sets. A *directed path* in a digraph $G = (W, A)$ is a vertex sequence (v_1, v_2, \dots, v_t) such that $(v_i, v_{i+1}) \in A$ for every $i = 1, 2, \dots, t-1$. We say that this is a path from v_1 to v_t , or simply a $(v_1 \rightarrow v_t)$ -path. For a digraph $G = (W, A)$, the *subgraph induced* by a $W' \subseteq W$, denoted by $G[W']$, is the digraph with vertex set W' and arc set $\{(a, b) \mid a, b \in W', (a, b) \in A\}$. Unless stated otherwise, in this paper we simply use “graph” for “undirected graph”.

Three NP-hard Problems. We assume familiarity with basic notation in complexity theory such as NP-hardness. Our NP-hardness results are shown by reductions from the following NP-hard problems.

EXACT 3 SET COVER (X3C)

Input: A universal set X with $|X| = 3\kappa$ for some positive integer κ and a collection \mathcal{C} of 3-subsets of X .

Question: Is there a subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $|\mathcal{C}'| = \kappa$ and each $x \in X$ appears in exactly one set of \mathcal{C}' ?

The NP-hardness of the X3C problem was given in [13]. In this paper, we assume that each element $x \in X$ appears in exactly three different 3-subsets of X in \mathcal{C} . Therefore, we have that $|\mathcal{C}| = 3\kappa$. This assumption does not change the NP-hardness of the X3C problem [14].

LABELED RED-BLUE DOMINATING SET (LRBDS)

Input: A bipartite graph $B = (R \uplus B, E)$, where each vertex in R has a label from $\{1, 2, \dots, k\}$.

Question: Is there a subset $W \subseteq R$ such that $|W \cap R_i| \leq 1$ for every $i \in \{1, 2, \dots, k\}$ and W dominates B , where R_i is the set of all vertices in R that has label i ?

Lemma 1. *The LRBDS problem is NP-hard.*

The proof for the above lemma is deferred to Appendix.

A *Boolean variable* x takes either the value 1 or 0. Let X be a set of Boolean variables. If $x \in X$, then x and \bar{x} are *literals* over X . A *clause* c over X is a set of literals over X . A *truth assignment* is a function $\varrho : X \mapsto \{0, 1\}$. A clause c is *satisfied* under a truth assignment ϱ if and only if there is an $x \in c$ such that $\varrho(x) = 1$, or a $\bar{x} \in c$ such that $\varrho(x) = 0$. The 3-Satisfiability problem defined below is a famous NP-hard problem [13].

3-SATISFIABILITY (3-SAT)

Input: A set X of boolean variables, and a collection C of clauses over X such that each clause includes exactly three literals.

Question: Is there a truth assignment $\varrho : X \mapsto \{0, 1\}$ under which all clauses in C are satisfied?

Parameterized Complexity. Parameterized complexity was introduced by Downey and Fellows [7] as a tool to deal with hard problems. A *parameterized problem* is a language $\Sigma^* \times \Sigma^*$, where Σ is a finite alphabet. The first component is called the *main part* of the problem and the second component is called the *parameter*. In this paper, we consider only positive integer parameters. A parameterized problem is *fixed-parameter tractable* (FPT) if it is solvable in $O(f(k) \cdot |I|^{O(1)})$ time, where I is the main part of the instance, k is the parameter, and $f(k)$ is a computable function depending only on k . For further discussion on parameterized complexity, we refer to the textbooks [8, 22].

1.2 Related Work

To the best of our knowledge, group identification as a classic model for identifying socially qualified individuals has not been studied from the complexity point of view. The words “control by adding/deleting/partition of” in the problem names is reminiscent of many strategic voting problems, such as control by adding/deleting/partition of voters/candidates, which have been extensively studied in the literature [9, 10, 15, 25, 26]. In a voting system, we have a set of candidates and a set of voters. Each voter casts a vote, and a voting correspondence is used to select a subset of candidates. From this standpoint, group identification can be considered as a voting system where the individuals are both voters and candidates. Nevertheless, group identification differs from voting systems in many significant aspects. First, the goal of a voting system is to select a subset of candidates, which are often called winners since they are considered as more competitive or outstanding compared with the remaining candidates for some specific purpose. Despite that the goal of group identification is also to identify a set of individuals (socially qualified individuals) from the whole individuals, it does not imply that socially qualified individuals are more competitive or outstanding than the remaining individuals. For instance, in situations where we want to identify left-wing party members among a group of people, the model of group identification is more suitable. In other words, group identification is more close to a classification model. Second, as voting systems aim to select a subset of competitive candidates for some special purpose, more often than not, the number of winners are pre-decided (e.g., in a single-winner voting, exactly one candidate is selected as the winner). As a consequence, many voting systems need to adopt a certain tie-breaking method to break the tie when many candidates are considered equally competitive. However, group identification does not need a tie breaking method, since there is no size bound of the number of socially qualified individuals.

It is also worth pointing out that the classic voting system Approval, which has been widely studied in the literature [3, 11, 17, 19, 24], has the flavor of group identification. In an Approval voting, each voter approves or disapproves each candidate. Thus, each voter’s vote is represented by an 1-0 vector, where the entries with 1s (resp. 0s) mean that the voter approves (resp. disapproves) the corresponding candidate. The winners are among the candidates which get the most approvals. If the voters and candidates are the same group of individuals, then it seems that Approval voting is a social rule. Nevertheless, as discussed above, Approval voting is more often considered as a single-winner voting system and thus need to utilize a tie breaking method. Recently, several variants of Approval voting have been studied as multi-winner voting systems. However, the number of winners is bounded by (or exactly equals to) an integer k [1]. Moreover, to the best of our knowledge, complexity of control by adding/deleting/partition of voters/candidates has not been studied for Approval voting when the voters and candidates coincide, though it is fairly easy to check that many complexity results in this case can be directly obtained from the results in the general case.

2 Complexity Results

In this section, we investigate GCAI, GCDI and GCPI for the liberal rule, consent rules, CSR rule and LSR rule. For each social rule, we study first if it is immune or susceptible to the problem under consideration. If it is susceptible, we further explore the complexity of the problem for the social rule. In particular, we achieve dichotomy results with respect to the consent quotas for the consent rules, as summarized in Table 1.

2.1 Consent Rules

Recall that the liberal rule is exactly the consent rule $f^{(1,1)}$ [23]. The intrinsic property of the liberal rule is that it completely leaves to each individual to determine whether herself is socially qualified or not. Put it another way, whether an individual is socially qualified is independent of the opinions of any other individuals. As a consequence, the answer to the question whether an individual in S is socially qualified before and after adding/deleting/partition of individuals remains the same, as implied by the following theorem.

Theorem 1. *The liberal rule is immune to the GCAI, GCDI, and GCPI problems.*

Proof. Consider instances of the GCAI, GCDI, and GCPI problems with the liberal rule as their social rule, i.e., $f \equiv f^L$. As assumptions of instances, $S \not\subseteq f(\varphi, T)$ with $S \subseteq T$ being imposed in the GCAI problem and $S \not\subseteq f(\varphi, N)$ being imposed in both the GCDI and GCPI problems. According to the definition of the liberal rule, each of the above assumptions implies that there exists an individual $a \in S$ such that $\varphi(a, a) = 0$, and hence, $a \notin f^L(\varphi, V)$ for every $V \subseteq N$. It follows that $S \not\subseteq f^L(\varphi, V)$ for every $S \subseteq V \subseteq N$. Therefore, for all instances with the liberal rule as the social rule, the answers to the GCAI, GCDI, and GCPI problems are always “NO”. This completes the proof. \square

Now we study consent rules with one of the consent quotas equal to 1 and the other greater than 1. These consent rules have some flavor of the liberal rule. In particular, every consent rule $f^{(1,t)}$ with $t \geq 2$ (resp. $f^{(s,1)}$ with $s \geq 2$) is positive (resp. negative) liberal, in the sense that an individual’s qualification (resp. disqualification) of herself is sufficient to determine her social qualification, regardless of the opinions of any other individuals. We shall see that similar to the liberal rule, consent rules $f^{(1,t)}$ and $f^{(s,1)}$ are correspondingly immune to some problems studied in this paper, as summarized in the following theorem.

Theorem 2. *Every consent rule $f^{(s,1)}$ with $s \geq 2$ is immune to the GCDI and GCPI problems, and every consent rule $f^{(1,t)}$ with $t \geq 2$ is immune to the GCAI problem.*

Proof. We first consider consent rules $f^{(s,1)}$ with $s \geq 2$. Let $a \in S$ be an individual which is not socially qualified, i.e., $a \notin f^{(s,1)}(\varphi, N)$. We distinguish between two cases.

Case $\varphi(a, a) = 1$: There are at most $s - 1$ individuals in N qualifying individual a , i.e., $|\{a' \in N \mid \varphi(a', a) = 1\}| < s$, and thus $|\{a' \in V \mid \varphi(a', a) = 1\}| < s$ for every $V \subseteq N$. Therefore, it is impossible to make individual $a \in S$ socially qualified by deleting or partition of individuals.

Case $\varphi(a, a) = 0$: By definition, each social rule f satisfies $f(\varphi, V) \subseteq V$, and hence, $S \not\subseteq f^{(s,1)}(\varphi, V)$ if $S \not\subseteq V$; otherwise, when $S \subseteq V$, we have $|\{a' \in V \mid \varphi(a', a) = 0\}| \geq 1$ from $\varphi(a, a) = 0$, which implies that $a \notin f^{(s,1)}(\varphi, V)$. Hence, it is impossible to make individual a socially qualified by deleting or partition of individuals, i.e., $S \not\subseteq f^{(s,1)}(\varphi, V)$ for any $V \subseteq N$.

Therefore, for each instance with $f^{(s,1)}$ as its social rule, the answers to GCDI and GCPI are always “NO”.

Now we come to the consent rule $f^{(1,t)}$. Let $a \in S$ be an individual which is not socially qualified, that is $a \notin f^{(1,t)}(\varphi, T)$. This implies that $\varphi(a, a) = 0$ and, moreover, there are at least t individuals a' (including a) in T such that $\varphi(a', a) = 0$. Therefore, no matter which individuals the set U includes, there will be still at least t individuals $a' \in T \cup U$ such that $\varphi(a', a) = 0$, implying that a is still not socially qualified. \square

Now we study consent rules where none of the consent quotas is equal to 1. We shall see that these rules are susceptible to all the three problems GCAI, GCDI and GCPI. In addition, we investigate the complexity of GCAI, GCDI and GCPI for consent rules $f^{(s,t)}$ with $s \geq 2$ and $t \geq 2$. We first study GCDI for consent rules $f^{(s,2)}$ with $s \geq 1$. To show that every consent rule $f^{(s,2)}$ with $s \geq 1$ is not immune to the GCDI problem, we need only to give an instance where we can make all individuals in S socially qualified by deleting a limited number of individuals, given that not all individuals in S are socially qualified in advance. To this end, consider an instance $(f^{(s,2)}, N = \{a, b\}, \varphi, S = \{a\}, k = 1)$ where $\varphi(a, a) = \varphi(b, a) = 0$. It is clear that we can make a socially qualified by deleting b from the instance. Now we study the complexity of the problem.

Theorem 3. *GCDI for every consent rule $f^{(s,2)}$ with $s \geq 1$ is polynomial-time solvable.*

Proof. Let $L = \{a \in S \mid \varphi(a, a) = 1\}$ and $\bar{L} = S \setminus L = \{a \in S \mid \varphi(a, a) = 0\}$. For each $a \in \bar{L}$, let $U_a \subseteq N$ be the set of individuals each of which is outside of S and disqualifies a , i.e., $U_a = \{a' \in N \setminus S \mid \varphi(a', a) = 0\}$. Moreover, let $U = \bigcup_{a \in \bar{L}} U_a$. Then, the algorithm returns “NO” if $S \not\subseteq f^{(s,2)}(\varphi, N \setminus U)$ or $|U| > k$, and otherwise returns “YES”.

The correctness of the algorithm is shown based on the following observations. According to the consent rule $f^{(s,2)}$, $a \in \bar{L}$ is socially qualified if there is no further individual $a' \neq a$ such that $\varphi(a', a) = 0$. Therefore, in order to make $a \in \bar{L}$ socially qualified, all individuals $a' \in N \setminus S$ with $\varphi(a', a) = 0$ have to be deleted. This directly implies that all individuals in U , as defined above, have to be deleted.

Now let us consider $f^{(s,2)}(\varphi, N \setminus U)$. Suppose $S \not\subseteq f^{(s,2)}(\varphi, N \setminus U)$, and let $a \in S \setminus f^{(s,2)}(\varphi, N \setminus U)$. We distinguish between the following two cases.

Case $a \in L$: According to the consent rule $f^{(s,2)}$, there are at most $s - 1$ individuals $a' \in N \setminus U$ such that $\varphi(a', a) = 1$. Since deleting individuals does not increase the number of individuals that qualify a , the individual a cannot be socially qualified; and thus, the given instance is a NO-instance.

Case $a \in \bar{L}$: In this case, there is an individual $a' \in S$ such that $a' \neq a$ and $\varphi(a', a) = 0$. Since we cannot delete individuals in S due to the definition of the problem, individual a cannot be socially qualified; and thus, the given instance is a NO-instance.

Due to the above analysis, if $S \not\subseteq f^{(s,2)}(\varphi, N \setminus U)$, we can safely return “NO”. Since we are allowed to delete at most k individuals, and according to the above analysis all individuals in U must be deleted, if $|U| > k$, we can safely return “NO” too. On the other hand, if $S \subseteq f^{(s,2)}(\varphi, N \setminus U)$ and $|U| \leq k$, U itself is an evidence for answering “YES”.

Finally, observe that the construction of the set U , and the decisions of $S \subseteq f^{(s,2)}(\varphi, N \setminus U)$ and $|U| \leq k$ can be done in $O(|N|^2)$ time. This completes the proof. \square

Theorem 3 reveals that it is practically tractable for an external agent to control a group identification procedure if the external agent is allowed to delete individuals and if the procedure adopts the consent rule $f^{(s,2)}$ to identify the socially qualified individuals.

Now we study GCAI for consent rules $f^{(s,t)}$ with $s \geq 2, t \geq 1$, and GCDI for consent rules $f^{(s,t)}$ with $s \geq 1, t \geq 3$. In contrast to the polynomial-time solvability of GCDI for consent rules $f^{(s,2)}$ with $s \geq 1$, as stated in Theorem 3, we prove that the same problem for consent rules with consent quotas $s \geq 1$ and $t \geq 3$ becomes NP-hard. In addition, we prove that GCAI for consent rules with consent quotas $s \geq 2$ and $t \geq 1$ is NP-hard too. Our results are summarized in the following theorem. It should be noted that the instances in our NP-hardness reductions directly imply that every consent rule $f^{(s,t)}$ with $s \geq 2$ and $t \geq 1$ is susceptible to the GCAI problem, and every consent rule $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$ is susceptible to the GCDI problem.

Theorem 4. *GCAI for every consent rule $f^{(s,t)}$ with $s \geq 2$ and $t \geq 1$, and GCDI for every consent rule $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$ are NP-hard.*

Proof. We prove the theorem by reductions from the X3C problem. Let's first consider GCAI for consent rules $f^{(2,t)}$ with $t \geq 1$. Given an instance $\mathcal{I} = (X, \mathcal{C})$ with $|X| = 3\kappa$, we create an instance $\mathcal{E}_{\mathcal{I}} = (f^{(s,t)}, N, \varphi, S, T, k)$ for the GCAI problem as follows.

There are $|X| + |\mathcal{C}|$ individuals in $N = \{a_x \mid x \in X\} \cup \{a_c \mid c \in \mathcal{C}\}$. The first $|X|$ individuals $\{a_x \mid x \in X\}$ one-to-one correspond to the elements in X , and the last $|\mathcal{C}|$ individuals $\{a_c \mid c \in \mathcal{C}\}$ one-to-one correspond to elements in \mathcal{C} . We define $S = T = \{a_x \in N \mid x \in X\}$. In addition, we set $k = \kappa$. Now we define the profile φ .

- For each $x, x' \in X$, $\varphi(a_x, a_{x'}) = 1$ if and only if $x = x'$.
- For each $x \in X$ and for each $c \in \mathcal{C}$, $\varphi(a_c, a_x) = 1$ if and only if $x \in c$.
- For each $c, c' \in \mathcal{C}$, $\varphi(a_c, a_{c'}) = 0$.

For the proof, the values of $\varphi(a_x, a_c)$ where $x \in X$ and $c \in \mathcal{C}$ are not essential. Obviously, the construction of $\mathcal{E}_{\mathcal{I}}$ can be done in polynomial time, namely $O((|X| + |\mathcal{C}|)^2)$ time.

Now we prove the correctness of the reduction, i.e., we show that \mathcal{I} is a YES-instance for X3C if and only if $\mathcal{E}_{\mathcal{I}}$ is a YES-instance for the GCAI problem.

(\Rightarrow ;) Suppose \mathcal{I} is a YES-instance for X3C, and let $\mathcal{C}' \subseteq \mathcal{C}$ be an exact 3-set cover, i.e., $|\mathcal{C}'| = k$ and for every $x \in X$ there exists a $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \in N \mid c \in \mathcal{C}'\}$. Then, according to the definition of φ , for each $a_x \in S$, there exists an $a_c \in U$ such that $\varphi(a_c, a_x) = 1$. Moreover, each $a_x \in S$ qualifies herself (i.e., $\varphi(a_x, a_x) = 1$). Therefore, according to the definition of the consent rule $f^{(2,t)}$, $a_x \in f^{(2,t)}(\varphi, T \cup U)$ for every $a_x \in S$, i.e., $S \subseteq f^{(2,t)}(\varphi, T \cup U)$. By definition, we have $|U| = |\mathcal{C}'| = k = \kappa$. Therefore, $\mathcal{E}_{\mathcal{I}}$ is a YES-instance for the GCAI problem.

(\Leftarrow ;) Suppose $\mathcal{E}_{\mathcal{I}}$ is a YES-instance for the GCAI problem, and let $U \subseteq N \setminus T$ be a set of individuals such that $|U| \leq k = \kappa$ and $S \subseteq f^{(2,t)}(\varphi, T \cup U)$. From $S \subseteq f^{(2,t)}(\varphi, T \cup U)$ and, for all $a_x, a_{x'} \in S = T$, $\varphi(a_x, a_{x'}) = 1$ if and only if $x = x'$, it follows that, for each $a_x \in S$, there is an $a_c \in U$ such that $\varphi(a_c, a_x) = 1$. Then, according to the definition of the profile φ , for each $x \in X$, there exists $c \in \mathcal{C}$ such that $a_c \in U$ and $x \in c$. This implies that $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U\}$ is an exact 3-set cover of \mathcal{I} . Thus, \mathcal{I} is a YES-instance.

The NP-hardness reduction for GCAI for any consent rule $f^{(s,t)}$ with $s > 2$ and $t \geq 1$ can be adapted from the above reduction. Precisely, we introduce further $s - 2$ dummy individuals in T , and let all these dummy individuals qualify every individual in $S = \{a_x \in N \mid x \in X\}$. The opinions of a dummy individual over any other individual in N and the other way around do not matter in the proof, and thus can be set arbitrarily. Now for each individual $a_x \in S$, there are exactly $s - 1$ individuals in T who qualify a_x . Moreover, in order to make each $a_x \in S$ socially qualified, we need one more individual in $N \setminus T$ who qualifies a_x .

Now let's consider GCDI for consent rules $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$. We first consider the case $t = 3$. The reduction for this problem is similar to the above reduction for GCAI for consent rules $f^{(2,t)}$ where $t \geq 1$ with the following differences.

1. There is no T in this reduction; but keeping $S = \{a_x \in N \mid x \in X\}$;
2. The values of $\varphi(a, b)$ for every $a, b \in N$ is reversed. That is, we have $\varphi(a, b) = 1$ in the current reduction if and only if $\varphi(a, b) = 0$ in the above reduction for the GCAI problem; and
3. $k = 2\kappa$.

Now we prove the correctness of the reduction.

(\Rightarrow ;) Suppose that there is an exact 3-set cover $\mathcal{C}' \subset \mathcal{C}$ for \mathcal{I} , i.e., $|\mathcal{C}'| = k$ and for every $x \in X$ there exists exactly one $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \mid c \in \mathcal{C} \setminus \mathcal{C}'\}$ and $U' = \{a_c \mid c \in \mathcal{C}'\}$. Clearly, $S \cap U = \emptyset$. Moreover, $N \setminus U = S \cup U'$. Let a_x be an individual in S where $x \in X$. Then, according to the construction, there is exactly one $a_c \in U'$ such that $\varphi(a_c, a_x) = 0$. Since $\varphi(a_{x'}, a_x) = 1$ for all $a_{x'} \in S \setminus \{a_x\}$, according to the consent rule $f^{(s,3)}$, $a_x \in f^{(s,3)}(\varphi, N \setminus U)$. Since this holds for every $a_x \in S$, we can conclude that $S \subseteq f^{(s,3)}(\varphi, N \setminus U)$.

(\Leftarrow ;) Suppose that there is a $U \subseteq N \setminus S$ such that $|U| \leq 2\kappa$ and $S \subseteq f^{(s,3)}(\varphi, N \setminus U)$. Let $U' = N \setminus (S \cup U)$, and $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U'\}$. Thus, $N \setminus U = S \cup U'$. Due to the fact $\varphi(a_x, a_x) = 0$ for every $a_x \in S$ where $x \in X$ and the definition of φ , it holds that for every $a_x \in S$, there is at most one $a_c \in U'$ such that $\varphi(a_c, a_x) = 0$ and $x \in c$. As a result, there is no $x \in X$ which belongs to two distinct sets in U' . Moreover, since $|U'| = 3\kappa - |U| \geq \kappa$, and $|S| = 3\kappa$, it follows that $|U'| = \kappa = k$ and $|\mathcal{C}'|$ is an exact 3-set cover of \mathcal{I} .

The NP-hardness of GCDI for any consent rule $f^{(s,t)}$ with $s \geq 1$ and $t > 3$ can be adapted from the above reduction by introducing some dummy individuals. In particular, we introduce further $t - 3$ individuals in S . Let S' denote the set of the $t - 3$ dummy individuals. Thus, $S = \{a_x \in N \mid x \in X\} \cup S'$. We want each dummy individual in S' to be a robust socially qualified individual, that is, every $d \in S'$ is socially qualified regardless of which individuals (at most $k = 2\kappa$) would be deleted. To this end, for every $d \in S'$, we let d disqualify herself, and let all the other individuals qualify d . We set $\varphi(d, a_x) = 0$ for every $d \in S'$ and a_x where $x \in X$. Thus, for every $a_x \in S$ where $x \in X$, there are in total $t + 1$ individuals in N who disqualify a_x . The other entries in the profile not defined above can be set arbitrarily. In order to make each $a_x \in S$ where $x \in X$ socially qualified, we need to delete exactly two individuals in $N \setminus S$ who disqualify a_x . This happens if and only if there is an exact 3-set cover for \mathcal{I} , as we discussed in the proof for the consent rule $f^{(s,3)}$. \square

Even though every consent rule $f^{(s,t)}$ with $s \geq 2$ and $t \geq 1$ (resp. $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$) is susceptible to the GCAI (resp. GCDI) problem, Theorem 4 reveals that it is unpractical for an external agent to successfully control a group

	qualified by	disqualified by
$a(x, 1)$	$N \setminus \{a(x, 1), a(x, 2)\}$	$a(x, 1), a(x, 2)$
$a(x, 2)$	$N \setminus (\{a(x, 2), a(C)\} \cup \{a(c) \mid c \in C\})$	$a(x, 2), a(C)$, and $a(c)$ for each $c \in C$
$a(C)$	$N \setminus (\{a(c) \mid c \in C\} \cup \{a(C)\})$	$a(C)$ and $a(c)$ for every $c \in C$
$a(c)$	$N \setminus (\{a(x, 2) \mid x \in c\} \cup \{a(x, 1) \mid \bar{x} \in c\} \cup \{a(c)\})$	$a(c)$, $a(x, 2)$ for every $x \in c$, $a(x, 1)$ for every $\bar{x} \in c$

Table 2: This table summarizes, for each individual $a \in N$, the set of individuals qualifying a and the set of individuals disqualifying a , according to the profile in the proof of Theorem 5.

identification procedure with every consent rule $f^{(s,t)}$ with $s \geq 2$ and $t \geq 1$ (resp. $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$) as the social rule to identify the socially qualified individuals, by adding (resp. deleting) individuals.

Now we study the GCPI problem. We have shown in Theorem 2 that all consent rules $f^{(s,t)}$ where $t = 1$ are immune to GCPI. We show now that if the consent quota $t > 1$, then the consent rule is susceptible to GCPI. Consider an instance $(f^{(s,t)}, N, \varphi, S)$ where $t \geq 2$, $N = \{a_1, a_2, \dots, a_{t+1}\}$ and $S = \{a_1\}$. Moreover, $\varphi(a_i, a_j) = 0$ for every $i, j \in \{1, 2, \dots, t+1\}$. Clearly, $f^{(s,t)}(\varphi, N) = \emptyset$. Now consider the partition $(U = S, N \setminus U)$ of N . Then, $f^{(s,t)}(\varphi, U) = S$. Moreover, for every individual $a_i \in N \setminus U$, at least t individuals in $N \setminus U$ disqualifying a_i , implying that $f^{(s,t)}(\varphi, N \setminus U) = \emptyset$. In summary, $S = \{a_1\} = f^{(s,t)}(\varphi, f^{(s,t)}(\varphi, U) \cup f^{(s,t)}(\varphi, N \setminus U))$. Now it is of particular interest to study the complexity of GCPI for consent rules.

Theorem 5. *GCPI for every consent rule $f^{(s,t)}$ with $s \geq 1$ and $t \geq 2$ is NP-hard.*

Proof. We prove the NP-hardness of the problem stated in the theorem by a reduction from the 3-SAT problem. We first consider GCPI for consent rules $f^{(s,2)}$ with $s \geq 1$. Later, we extend the reduction to all consent rules $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$.

Let (X, C) be an instance of the 3-SAT problem, where X is the set of Boolean variables and C is the set of clauses each consisting of three literals. Moreover, let m and n be the numbers of variables and clauses, respectively, i.e., $m = |X|$ and $n = |C|$. We construct an instance $\mathcal{E} = (f^{(s,2)}, N, \varphi, S)$ of GCPI for $f^{(s,2)}$ as follows.

There are in total $2m + n + 1$ individuals in N . In particular, for each variable $x \in X$, we create two individuals $a(x, 1)$ and $a(x, 2)$. In addition, for each clause $c \in C$, we create one individual $a(c)$. In addition, we create one individual $a(C)$ for C . The set $S = \{a(x, 1) \mid x \in X\} \cup \{a(C)\}$. The profile is defined as follows.

1. for each $a \in N$, $\varphi(a, a) = 0$.
2. for each $x \in X$, $\varphi(a(x, 2), a(x, 1)) = 0$.
3. for each $c \in C$, $\varphi(a(c), a(C)) = 0$.
4. for each $x \in X$, $\varphi(a(C), a(x, 2)) = 0$.
5. for each $x \in X$ and $c \in C$, $\varphi(a(c), a(x, 2)) = 0$.
6. for each $c \in C$ and every variable x involved in c , $\varphi(a(x, 2), a(c)) = 0$ if $x \in c$ and $\varphi(a(x, 1), a(c)) = 0$ if $\bar{x} \in c$.
7. for every two $a, a' \in N$ such that $\varphi(a, a')$ is not defined above, $\varphi(a, a') = 1$.

Now we prove that (X, C) is a YES-instance if and only if \mathcal{E} is a YES-instance. Table 2 is helpful for the reader to check the following arguments.

(\Rightarrow .) Assume that there is a truth assignment $\varrho : X \mapsto \{0, 1\}$. Then, we find an $U \subseteq N$ as follows. First, U include the individual $a(C)$ and exclude all individuals in $\{a(c) \mid c \in C\}$, i.e., $a(C) \in U$ and $\{a(c) \mid c \in C\} \subseteq N \setminus U$. In addition, for each $x \in X$, U includes exactly one of $\{a(x, 1), a(x, 2)\}$, depending on the value of $\varrho(x)$. In particular, for every $x \in X$, $a(x, 1) \in U$ and $a(x, 2) \in N \setminus U$ if $\varrho(x) = 1$; and $a(x, 2) \in U$ and $a(x, 1) \in N \setminus U$; otherwise. Now let's consider the subprofiles $f^{(s,2)}(\varphi, U)$ and $f^{(s,2)}(\varphi, N \setminus U)$. Since the only individual in U that disqualifies $a(C)$ is $a(C)$ herself, it holds that $a(C) \in f^{(s,2)}(\varphi, U)$. Let x be any variable in X . Due to the above definition of U , $a(x, 2)$ is not partitioned in the same set with $a(x, 1)$. Since the only individuals that disqualify $a(x, 1)$ are $a(x, 1)$ and $a(x, 2)$, it holds that $a(x, 1)$ survives the first stage of selection, i.e., $a(x, 1) \in f^{(s,2)}(\varphi, U)$ if $a(x, 1) \in U$ and $a(x, 1) \in f^{(s,2)}(\varphi, N \setminus U)$ if $a(x, 1) \in N \setminus U$. On the other hand, since $a(C)$ and all individuals in $\{a(c) \mid c \in C\}$ disqualify every individual in $\{a(x, 2) \mid x \in X\}$, none of $\{a(x, 2) \mid x \in X\}$ survives the first stage of selection, i.e., for every $x \in X$ it holds that $a(x, 2) \notin f^{(s,2)}(\varphi, W)$ where $W \in \{U, N \setminus U\}$ and $a(s, 2) \in W$. Now we consider the individuals corresponding to the clauses. Let $c \in C$ be a clause. Since c is satisfied under ϱ , there is either an $x \in c$ such that $\varrho(x) = 1$, or a $\bar{x} \in c$ such that $\varrho(x) = 0$. In the former case, we have $a(x, 2) \in N \setminus U$, $\varphi(a(x, 2), a(c)) = 0$, and in the latter case we have $a(x, 1) \in N \setminus U$, $\varphi(a(x, 1), a(c)) = 0$. Hence, both cases lead $a(c)$ to be eliminated in the first stage of selection, i.e., $\{a(c) \mid c \in C\} \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. As a summary,

$f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U) = S$. Since every individual in S is only disqualified by herself, i.e., $\varphi(a', a) = 0$ if and only if $a' = a$ for every $a, a' \in S$, we have that $S = f^{(s,2)}(\varphi, S)$. This completes the proof of this direction.

(\Leftarrow) Suppose that there is a $U \subseteq N$ such that $S \subseteq f^{(s,2)}(\varphi, f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U))$. Due to symmetry, assume that $a(C) \in U$. Since for every $c \in C$, $\varphi(a(c), a(C)) = 0$, and $a(C) \in S$, it holds that $\{a(c) \mid c \in C\} \subseteq N \setminus U$ (otherwise, $a(C)$ would be eliminated in the subprofile restricted to U). Moreover, it holds that $\{a(c) \mid c \in C\} \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. As a result, for every $a(c)$, except herself, there must be at least one other individual in $N \setminus U$ that disqualifies $a(c)$. Due to the definition of the profile, this means that there is either an $x \in c$ such that $a(x, 2) \in N \setminus U$, or a $\bar{x} \in c$ such that $a(x, 1) \in N \setminus U$. Since for every $x \in X$ it holds $\varphi(a(x, 2), a(x, 1)) = \varphi(a(x, 1), a(x, 2)) = 0$, exactly one of $\{a(x, 1), a(x, 2)\}$ can be in $N \setminus U$ (otherwise, $a(x, 1)$ would be eliminated in the first stage of selection). Hence, given U , we can uniquely define a truth assignment ϱ as follows. For every $x \in X$, define $\varrho(x) = 1$ if $a(x, 2) \in N \setminus U$ and $a(x) = 0$ otherwise. Then, due to the above discussion, for every $c \in C$, there is either an $x \in c$ such that $\varrho(x) = 1$ or a $\bar{x} \in c$ such that $\varrho(x) = 0$. Therefore, every clause is satisfied under the truth assignment ϱ . This completes the proof of this direction.

Now, we explain how to extend the above reduction for each consent rule $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$. Assume that $|C| \geq t - 1$ (if this is not the case, we can duplicate any arbitrary clause to make the inequality hold). In addition to the individuals defined in the above reduction, we further create $2t - 4$ dummy individuals $a_1^1, \dots, a_1^{t-2}, a_2^1, \dots, a_2^{t-2}$, and add all individuals a_1^1, \dots, a_1^{t-2} in S . Let $A_1 = \{a_1^1, \dots, a_1^{t-2}\}$ and $A_2 = \{a_2^1, \dots, a_2^{t-2}\}$. So we have now $S = \{a(x, 1) \mid x \in X\} \cup a(C) \cup A_1$, and

$$N = S \cup A_2 \cup \{a(x, 2) \mid x \in X\} \cup \{a(c) \mid c \in C\}.$$

Every individual in $A_1 \cup A_2$ is disqualified by all individuals in $A_1 \cup A_2$. In addition, all individuals in $N \setminus (A_1 \cup A_2)$ disqualify every individual in A_2 , and all individuals in $N \setminus (A_1 \cup A_2 \cup \{a(c) \mid c \in C\})$ qualify every individual in A_1 . Furthermore, all individuals in $\{a(c) \mid c \in C\}$ disqualify all individuals in A_1 . Finally, all individuals in $A_1 \cup A_2$ disqualify all individuals not in $A_1 \cup A_2$. The subprofile restricted to individuals not in $A_1 \cup A_2$ remains unchanged. Observe that, by defining so, to make all individuals in A_1 socially qualified, all dummy individuals must be balanced partitioned, i.e., the numbers of dummy individuals in U and $N \setminus U$ are both equal to $t - 2$, for every solution U . Assume for the sake of contradiction that this is not the case. Let $U \subseteq N$ be a solution, and due to symmetry, assume that $|U \cap (A_1 \cup A_2)| \geq t - 1$. Apparently, U includes at least one individual $a_1^i \in S \cap A_1$. Moreover, all individuals in $\{a(x, 1) \mid x \in X\} \cup \{a(C)\}$, all of which are in S , must be in the set $N \setminus U$, since otherwise, all of them will be eliminated in the first stage of selection, i.e., for every $a \in \{a(x, 1) \mid x \in X\} \cup \{a(C)\}$ it holds that $a \notin f^{(s,t)}(\varphi, W)$ where $W \in \{U, N \setminus U\}$ and $a \in W$. As a result, at most $t - 2$ individuals in $\{a(c) \mid c \in C\}$ can be included in $N \setminus U$, since otherwise, the individual $a(C)$ will be eliminated, i.e., $a(C) \notin f^{(s,t)}(\varphi, N \setminus U)$. As $|C| \geq t - 1$, there will be at least one individual $a(c)$ where $c \in C$ in the set U . However, the individual $a(c)$ together with all other individuals in $U \cap (A_1 \cup A_2)$, will make a_1^i be eliminated, contradicting that U is a solution. The observation follows. The discussion for the observation also implies that for every solution $U \subseteq N$, either $A_1 \subseteq U$ or $A_1 \subseteq N \setminus U$, i.e., all individuals in A_1 must be partitioned into the same set. Now, one can check that from every solution U of the instance constructed above for the consent rule $f^{(s,2)}$, we can get a solution for the instance constructed for consent rule $f^{(s,t)}$ with $s \geq 1$ and $t \geq 3$ by adding all individuals A_1 in U or $N \setminus U$ (if $a(C) \in U$ then $A_1 \subseteq U$; otherwise $A_1 \subseteq N \setminus U$), and vice versa. This completes the proof for the theorem. \square

2.2 Procedure Rules

In this section, we study the LSR and the CSR social rules. We first prove that the GCAI problem is NP-hard for both the LSR and the CSR social rules. The instances created in the proof of the following theorem directly imply that both the LSR and the CSR social rules are susceptible to the GCAI problem.

Theorem 6. *GCAI for both f^{LSR} and f^{CSR} are NP-hard.*

Proof. We prove the theorem by reductions from the X3C problem. Let's first consider the social rule f^{LSR} . Given an instance $\mathcal{I} = (X, C)$ with $|X| = 3\kappa$, we create an instance $\mathcal{E}_{\mathcal{I}} = (f^{LSR}, N, \varphi, S, T, k)$ for the GCAI problem as follows.

The definitions of N, S, T and k are the same as in the NP-hardness reduction for GCAI for consent rules $f^{(2,t)}$ with $t \geq 1$ in Theorem 4. The profile φ is defined as follows.

1. For each $x, x' \in X$, $\varphi(a_x, a_{x'}) = 0$.
2. For each $x \in X$ and each $c \in C$, $\varphi(a_x, a_c) = 0$.
3. For each $c, c' \in C$, $\varphi(a_c, a_{c'}) = 1$ if and only if $c = c'$.
4. For each $x \in X$ and each $c \in C$, $\varphi(a_c, a_x) = 1$ if and only if $x \in c$.

Now we prove the correctness of the reduction.

(\Rightarrow) Suppose that there is an exact 3-set cover $C' \subset C$ for \mathcal{I} , i.e., $|C'| = k$ and for every $x \in X$ there exists exactly one $c \in C'$ such that $x \in c$. Let $U = \{a_c \mid c \in C'\}$. According to the definition of φ , it holds that $U \subseteq f^{LSR}(\varphi, T \cup U)$. Moreover, for every $a_x \in S$ where $x \in X$, there is an $a_c \in U$ such that $\varphi(a_c, a_x) = 1$ and $x \in c$. Since $U \subseteq f^{LSR}(\varphi, T \cup U)$, according to the definition of the social rule f^{LSR} , it holds that $a_x \in f^{LSR}(\varphi, T \cup U)$ for every $a_x \in S$. Therefore, $\mathcal{E}_{\mathcal{I}}$ is a YES-instance since it has a solution U .

(\Leftarrow): Suppose that there is a $U \subseteq N \setminus T$ such that $|U| \leq k$ and $S = T \subseteq f^{LSR}(\varphi, T \cup U)$. Let $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U\}$. According to the definition of φ , $f^{LSR}(\varphi, T) = \emptyset$. Moreover, every $a_x \in S$ where $x \in X$ disqualifies all individuals in N , and every $a_c \in N \setminus T$ qualifies herself. As a result, for every $a_x \in S$ where $x \in X$, there must be at least one $a_c \in U$ where $c \in \mathcal{C}'$ such that $\varphi(a_c, a_x) = 1$. According to the definition of φ , this implies that for every $x \in X$, there is at least one $c \in \mathcal{C}'$ such that $x \in c$. Since $|\mathcal{C}'| = |U| \leq k = \kappa$, this implies that $|\mathcal{C}'| = k$ and, more precisely, \mathcal{C}' is an exact 3-set cover of \mathcal{I} .

Now let's consider GCAI for f^{CSR} . Again, the definitions of N, S, T and k are the same as in the NP-hardness reduction of GCAI for consent rules $f^{(2,t)}$ with $t \geq 1$ in Theorem 4. The profile φ is defined as follows.

- For each $x, x' \in X$, $\varphi(a_x, a_{x'}) = 0$.
- For each $x \in X$ and each $c \in \mathcal{C}$, $\varphi(a_x, a_c) = 1$.
- For each $c, c' \in \mathcal{C}$, $\varphi(a_c, a_{c'}) = 1$.
- For each $x \in X$ and each $c \in \mathcal{C}$, $\varphi(a_c, a_x) = 1$ if and only if $x \in c$.

Now we prove the correctness of the reduction.

(\Rightarrow): Suppose that there is a $\mathcal{C}' \subset \mathcal{C}$ such that $|\mathcal{C}'| = k$ and for every $x \in X$ there exists exactly one $c \in \mathcal{C}'$ such that $x \in c$. Let $U = \{a_c \mid c \in \mathcal{C}'\}$. Clearly, $|U| = |\mathcal{C}'| = k$. Observe that $U \subseteq f^{CSR}(\varphi, T \cup U)$. Then, according to the definition of φ , it holds that for every $a_x \in S$ where $x \in X$, there is an $a_c \in U$ such that $x \in c$ and $\varphi(a_c, a_x) = 1$. This implies that $a_x \in f^{CSR}(\varphi, T \cup U)$ for every $a_x \in S$. Thus, $\mathcal{E}_{\mathcal{I}}$ is a YES-instance, since it has a solution U .

(\Leftarrow): Suppose that there is a subset $U \subseteq N \setminus T$ such that $|U| \leq k$ and $S = T \subseteq f^{CSR}(\varphi, T \cup U)$. Let $\mathcal{C}' = \{c \in \mathcal{C} \mid a_c \in U\}$. According to the definition of φ , $f^{CSR}(\varphi, T) = \emptyset$. Moreover, every individual in S disqualifies every individual in S . Furthermore, every individual in $N \setminus T$ is qualified by all individuals in N . Therefore, for every $a_x \in S$ where $x \in X$, there must be at least one $a_c \in U$ such that $\varphi(a_c, a_x) = 1$. According to the definition of φ , this implies that for every $x \in X$ there is at least one $c \in \mathcal{C}'$ such that $x \in c$. Since $|\mathcal{C}'| = |U| \leq k = \kappa$, this implies that $|\mathcal{C}'| = k$ and, more precisely, \mathcal{C}' is an exact 3-set cover of \mathcal{I} . \square

Now we study GCDI and GCPI for f^{CSR} and f^{LSR} .

Theorem 7. *The social rules f^{CSR} and f^{LSR} are immune to the GCDI and GCPI problems.*

Proof. We first give alternative explanations of the social rules f^{CSR} and f^{LSR} from the graph theory point of view, respectively.

Let N be a set of individuals. For every profile φ over N , we define a directed graph $\mathcal{G}_N^\varphi = (V, A)$ as follows. The vertex set V is a copy of N . For each $a \in N$, let $v(a)$ denote the copy of a in V . The arcs in A are defined as follows: there is an arc $(v(a), v(a'))$ if and only if $\varphi(a, a') = 1$ for each $a, a' \in N$. Notice that there may exist loops in the digraph. Let T be a subset of individuals and let $V(T) = \{v(a) \mid a \in T\}$. For a vertex $v(a) \in V(T)$, let $\Gamma(v(a), T)$ be the set of all vertices $v(a') \in V(T)$ such that there is a $(v(a) \rightarrow v(a'))$ -path in the induced digraph $\mathcal{G}_N^\varphi[V(T)]$. Moreover, for $H \subseteq V(T)$, let $\Gamma(H, T) = \bigcup_{v(a) \in H} \Gamma(v(a), T)$. Let $K^{LSR}(T) = \{v(a) \in V(T) \mid (v(a), v(a)) \in A\}$ and $K^{CSR}(T) = \{v(a) \in V(T) \mid \forall a' \in T, (v(a'), v(a)) \in A\}$. Then,

$$f^{CSR}(\varphi, T) = \{a \in T \mid v(a) \in \Gamma(K^{CSR}(T), T)\};$$

$$f^{LSR}(\varphi, T) = \{a \in T \mid v(a) \in \Gamma(K^{LSR}(T), T)\}.$$

Notice that since $(v(a), v(a)) \in A$ for every $a \in N$ with $\varphi(a, a) = 1$, it holds that $K^{CSR}(T) \subseteq \Gamma(K^{CSR}(T), T)$ and $K^{LSR}(T) \subseteq \Gamma(K^{LSR}(T), T)$.

According to the above definition, an individual $a \in T \subseteq N$ is a socially qualified individual of T with respect to f^{CSR} (resp. f^{LSR}) if and only if there is an individual $a' \in T$ such that $v(a') \in K^{CSR}(T)$ (resp. $v(a') \in K^{LSR}(T)$) and there is a $(v(a'), v(a))$ -path in the digraph $\mathcal{G}_N^\varphi[V(T)]$. See Figure 1 for an example. Since a directed path exists in an induced subgraph of \mathcal{G}_N^φ only if this path exists in \mathcal{G}_N^φ , deleting individuals cannot make an individual which is not socially qualified socially qualified. It follows that the social rules f^{CSR} and f^{LSR} are immune to the GCDI problem. Since in the GCPI problem some individuals may be deleted in the first stage of selection, the social rules f^{CSR} and f^{LSR} are immune to the GCPI problem too. \square

3 Bounded Group Size and Parameterized Complexity

In this section, we investigate how the size of S affects the complexity of the problems studied in the previous section. We have shown in Theorems 4 and 5 that GCAI, GCDI and GCPI for consent rules $f^{(s,t)}$ are NP-hard when either s or t exceeds some constant. Hence, there are no polynomial-time algorithms for these problems unless $P=NP$. Nevertheless, in many real-world situations, the size of S , the set of individuals whom the external agent wants to make socially qualified, may be relatively small. For such situations, studying the parameterized complexity of the group control problems with respect to the size of S is of particular importance.

	a_1	a_2	a_3	a_4
a_1	1	0	1	1
a_2	0	0	0	1
a_3	0	0	0	1
a_4	0	1	0	1

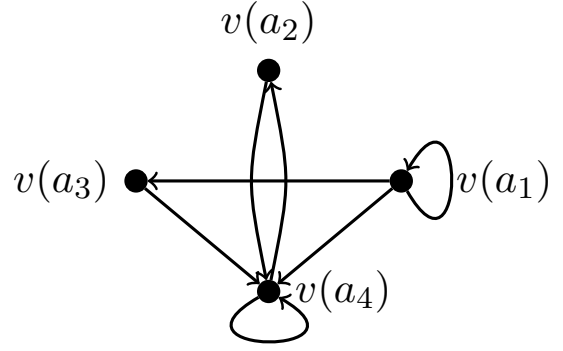


Figure 1: An illustration of the graph-based explanation of the social rules f^{CSR} and f^{LSR} . On the left side is a profile φ over the set $N = \{a_1, a_2, a_3, a_4\}$. On the right side is the digraph $\mathcal{G}_N^\varphi = (V, A)$. It holds that $K^{CSR}(N) = \{v(a_4)\}$ and $K^{LSR}(N) = \{v(a_1), v(a_4)\}$. Moreover, $\Gamma(K^{CSR}, N) = \{v(a_4), v(a_2)\}$ and $\Gamma(K^{LSR}, N) = \{v(a_1), v(a_4), v(a_2), v(a_3)\}$. Thus, $f^{CSR}(\varphi, N) = \{a_2, a_4\}$ and $f^{LSR}(\varphi, N) = \{a_1, a_2, a_3, a_4\}$.

We first study GCAI and GCDI for consent rules $f^{(s,t)}$. In particular, we prove that both GCAI and GCDI for consent rules are FPT with respect to the size of S . To this end, we give integer linear programming (ILP for short) formulations with the number of variables bounded by $2^{|S|}$ for both problems. As ILP is FPT with respect to the number of variables [12, 16, 18], so are GCAI and GCDI for consent rules.

Lemma 2 ([12, 16, 18]). *ILP can be solved using $O(v^{2.5v+o(v)} \cdot L)$ arithmetic operations, where L is the number of bits in the input and v is the number of variables in ILP.*

Now, we describe the ILP formulations for both GCAI and GCDI for consent rules.

Theorem 8. *GCAI and GCDI for every consent rule are FPT with respect to the size of S .*

Proof. We prove the theorem by giving ILP formulations for the GCAI and GCDI problems. The number of variables in the formulations is bounded by a function of $|S|$. We first consider the GCAI problem.

Let $(f^{(s,t)}, N, \varphi, S, T, k)$ be an instance of the GCAI problem. Let $\mu = |S|$. We say two individuals $a, b \in N$ have the *same opinion* over S , if for every $c \in S$, it holds that $\varphi(a, c) = \varphi(b, c)$. Hereinafter, let (a_1, a_2, \dots, a_n) be any arbitrary but fixed order of N . Let $S = \{a_{\lambda(1)}, a_{\lambda(2)}, \dots, a_{\lambda(\mu)}\}$ where $1 \leq \lambda(i) < \lambda(j) \leq n$ for every $1 \leq i < j \leq \mu$. For an individual $a_i \in N$ where $1 \leq i \leq n$, let $\varphi_{(a_i, S)}$ denote the vector $\langle \varphi(a_i, a_{\lambda(1)}), \varphi(a_i, a_{\lambda(2)}), \dots, \varphi(a_i, a_{\lambda(\mu)}) \rangle$.

The ILP formulation for the instance is as follows. For every μ -dimensional 1-0 vector β , let $N_\beta = \{a_i \in N \setminus T \mid \varphi_{(a_i, S)} = \beta\}$ and let $n_\beta = |N_\beta|$. We create a variable x_β for every μ -dimensional 1-0 vector β . Thus, there are in total 2^μ variables. Each variable x_β indicates how many individuals from N_β are included in the solution U . These variables are subject to the following restrictions. Let \mathfrak{V} be the set of all μ -dimensional 1-0 vectors.

(1) Since for every μ -dimensional 1-0 vector β there are at most n_β individuals $a_i \in N \setminus T$ such that $\varphi_{(a_i, S)} = \beta$, we need to ensure that no more than n_β of these individuals are in U . Moreover, every variable should be non-negative. Thus, every variable x_β is subject to

$$0 \leq x_\beta \leq n_\beta.$$

(2) Since we can add at most k individuals in total, it has to be that

$$\sum_{\beta \in \mathfrak{V}} x_\beta \leq k.$$

(3) In order to make every individual in S socially qualified, it has to be that

(3.1) for every $a_{\lambda(i)} \in S$ such that $\varphi(a_{\lambda(i)}, a_{\lambda(i)}) = 1$

$$\sum_{a_j \in T} \varphi(a_j, a_{\lambda(i)}) + \sum_{\beta \in \mathfrak{V}} (\beta[i] \cdot x_\beta) \geq s; \text{ and}$$

(3.2) for every $a_{\lambda(i)} \in S$ such that $\varphi(a_{\lambda(i)}, a_{\lambda(i)}) = 0$

$$\sum_{a_j \in T} (1 - \varphi(a_j, a_{\lambda(i)})) + \sum_{\beta \in \mathfrak{V}} ((1 - \beta[i]) \cdot x_\beta) \leq t - 1,$$

where $\beta[i]$ is the i -th component of β . The inequality (3.1) is to ensure that for every $a_{\lambda(i)} \in S$ who qualifies herself there are at least s individuals in the final profile who qualify $a_{\lambda(i)}$, and the inequality (3.2) is to ensure that for every individual $a_{\lambda(i)} \in S$ who disqualifies herself there are at most $t - 1$ individuals in the final profile who disqualify $a_{\lambda(i)}$.

Now let's consider the GCDI problem. Let $(f^{(s,t)}, N, \varphi, S, k)$ be a given instance of the GCDI problem. The ILP formulation for the instance is similar to the one for GCAI discussed above. Let (a_1, a_2, \dots, a_n) , $\{a_{\lambda(1)}, a_{\lambda(2)}, \dots, a_{\lambda(\mu)}\}$, $\varphi_{(a_i, S)}$, and

\mathfrak{V} be defined with the same meanings as above. For every μ -dimensional 1-0 vector β , let $\bar{N}_\beta = \{a_j \in N \setminus S \mid \varphi(a_j, S) = \beta\}$ and $\bar{n}_\beta = |\bar{N}_\beta|$. We create a variable y_β for every $\beta \in \mathfrak{V}$. Each variable y_β indicates how many individuals from \bar{N}_β are deleted. The restrictions are as follows.

(1) For every $\beta \in \mathfrak{V}$ we can delete at most \bar{n}_β individuals in \bar{N}_β . Moreover, each variable should be non-negative. Thus, for every variable y_β , we have that

$$0 \leq y_\beta \leq \bar{n}_\beta.$$

(2) Since we can delete at most k individuals in total, we have that

$$\sum_{\beta \in \mathfrak{V}} y_\beta \leq k.$$

(3) In order to make every individual in S socially qualified, it has to be that

(3.1) for every $a_{\lambda(i)} \in S$ such that $\varphi(a_{\lambda(i)}, a_{\lambda(i)}) = 1$

$$\sum_{a_j \in N} \varphi(a_j, a_{\lambda(i)}) - \sum_{\beta \in \mathfrak{V}} (\beta[i] \cdot y_\beta) \geq s; \text{ and}$$

(3.2) for every $a_{\lambda(i)} \in S$ such that $\varphi(a_{\lambda(i)}, a_{\lambda(i)}) = 0$

$$\sum_{a_j \in N} (1 - \varphi(a_j, a_{\lambda(i)})) - \sum_{\beta \in \mathfrak{V}} ((1 - \beta[i]) \cdot y_\beta) \leq t - 1.$$

According to Lemma 2, both ILPs shown above are solvable in time $O(v^{2.5v+o(v)} \cdot \text{poly}(v \cdot n))$, where $v = 2^\mu$. As a result, both GCAI and GCDI are FPT with respect to $\mu = |S|$. \square

Now, we study GCPI for consent rules $f^{(s,t)}$. In contrast to the fixed-parameter tractability of GCAI and GCDI for consent rules, we show that GCPI is unlikely to admit an FPT-algorithm. In particular, we prove that GCPI for consent rules $f^{(s,2)}$ such that $s \geq 3$ remains NP-hard even when S is a singleton. This directly implies that GCPI for consent rules is not FPT¹.

Theorem 9. *GCPI is NP-hard for consent rules $f^{(s,2)}$ such that $s \geq 3$, even when $|S| = 1$.*

Proof. We prove the theorem by a reduction from the LRBDS problem. Let $\mathcal{I} = (G = (R \uplus B, E), \{1, 2, \dots, k\})$ be an instance of the LRBDS problem. Let $s \geq 3$. We create an instance $\mathcal{E}_{\mathcal{I}} = (f^{(s,2)}, N, \varphi, S)$ for the GCPI problem for the consent rule $f^{(s,2)}$ as follows. We create $k + s - 2 + |B| + |R|$ individuals in total. Let (R_1, R_2, \dots, R_k) be the partition of R with respect to the labels of the vertices. That is, R_i where $1 \leq i \leq k$, is the set of vertices in R with label i . For each vertex $v \in R_i$ where $1 \leq i \leq k$, we create an individual $a_i(v)$. Let $A_i = \{a_i(v) \mid v \in R_i\}$. Moreover, for every vertex $u \in B$, we create an individual $a(u)$. Let $A(B) = \{a(u) \mid u \in B\}$. In addition, we create $k + 1$ individuals $C = \{c_1, c_2, \dots, c_k\} \cup \{w\}$, where each $c_i, 1 \leq i \leq k$, corresponds to the label i and $S = \{w\}$. Fianlly, we create $s - 3$ dummy individuals $A_{dummy} = \{d_1, d_2, \dots, d_{s-3}\}$. Hence, $N = \bigcup_{1 \leq i \leq k} A_i \cup A(B) \cup C \cup A_{dummy}$. The profile φ is defined as follows.

1. $\varphi(w, w) = 0$;
2. $\varphi(a(u), a(u')) = 0$ for every $u, u' \in B$ if and only if $u = u'$;
3. $\varphi(c_i, c_j) = 1$ for every $c_i, c_j \in C$ if and only if $i = j$;
4. $\varphi(x, w) = 0$ for every $x \in C \cup A(B)$;
5. $\varphi(a_i(v), w) = 1$ for every $v \in R_i$ where $1 \leq i \leq k$;
6. $\varphi(c_i, a(u)) = 1$ for every $c_i \in C$ and $a(u) \in A(B)$;
7. $\varphi(a(u), c_i) = 0$ for every $a(u) \in A(B)$ and $c_i \in C$;
8. $\varphi(d_i, d_{i'}) = 0$ for every dummy individual $d_i, d_{i'} \in A_{dummy}$;
9. $\varphi(d_i, w) = 0$ for every dummy individual $d_i \in A_{dummy}$;
10. $\varphi(d_i, x) = 1$ for every dummy individual $d_i \in A_{dummy}$ and every $x \in N \setminus (A_{dummy} \cup \{w\})$;
11. $\varphi(x, d_i) = 0$ for every dummy individual $d_i \in A_{dummy}$ and every $x \in N \setminus (A_{dummy} \cup \{w\})$;
12. $\varphi(a_i(v), a(u)) = 0$ for every $v \in R_i$ where $1 \leq i \leq k$ and every $a(u) \in A(B)$ if and only if $(v, u) \in E$;
13. $\varphi(a_i(v), c_j) = 1$ for every $a_i(v) \in R_i$ and $c_j \in C$ if and only if $i = j$; and

¹In fact, this further implies that GCPI for consent rules is even beyond XP, the class of all parameterized problems which are solvable in $O(|I|^{O(k)})$ time, where I is the main part and k is the parameter.

14. $\varphi(x, y)$ which is not defined above can be set arbitrarily.

Now we show the correctness of the reduction.

(\Rightarrow): Let W be a labeled red-blue dominating set of G . We shall show that \mathcal{E}_T is a YES-instance. Let $U \subseteq N$ be the set consisting of the individual w and all individuals that correspond to $R \setminus W$. That is, $U = S \cup \{a_i(v) \mid v \in R_i \setminus W, 1 \leq i \leq k\}$. Since $\varphi(w, w) = 0$, and every individual corresponding to some vertex in R qualifies w (see 5), it holds that $w \in f^{(s,2)}(\varphi, U)$. Now, let's consider the profile restricted to $N \setminus U$. Observe that $(N \setminus U) \cap (\bigcup_{1 \leq i \leq k} A_i) = \{a_i(v) \mid v \in R_i \cap W, 1 \leq i \leq k\}$. Let $a(u)$ be a candidate in $A(B)$ where $u \in B$. According to the construction of φ and the fact that W dominates B , there is at least one individual $a_i(v)$, corresponding to a vertex $v \in W$ dominating u , that disqualifies $a(u)$ (see 12). Since $\varphi(a(u), a(u)) = 0$ (see 2), it holds that $a(u) \notin f^{(s,2)}(\varphi, N \setminus U)$. Since this holds for every $a(u) \in A(B)$, we have that $A(B) \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. On the other hand, for every $1 \leq i \leq k$, since $|W \cap R_i| \leq 1$, $N \setminus U$ contains at most one individual $a_i(v) \in A_i$. According to the construction of φ , for every $c_i \in C$ only the following $s - 2$ individuals in $N \setminus U$ qualify c_i : (1) c_i herself; (2) $a_i(v) \in A_i$ where $v \in W$ (see 13); and (3) all $s - 3$ dummy individuals (see 10). It directly follows that $c_i \notin f^{(s,2)}(\varphi, N \setminus U)$ for every $c_i \in C$. Finally, since $\varphi(d_i, d_{i'}) = 0$ for every $d_i, d_{i'} \in A_{dummy}$ and all individuals in $N \setminus U$ disqualify all dummy individuals, it holds that $d_i \notin f^{(s,2)}(\varphi, N \setminus U)$ for every $d_i \in A_{dummy}$. In conclusion, $(A(B) \cup C \cup A_{dummy}) \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. Now, it is easy to verify that $\varphi(x, w) = 1$ for every $x \in f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U) \setminus \{w\}$. As a result, $w \in f^{(s,2)}(\varphi, f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U))$.

(\Leftarrow): Let $U \subseteq N$ such that $w \in f^{(s,2)}(\varphi, f^{(s,2)}(\varphi, U) \cup f^{(s,2)}(\varphi, N \setminus U))$. Due to symmetry, assume that $w \in U$. Since $\varphi(w, w) = 0$, all the other individuals that disqualify w must be in $N \setminus U$. That is, $A(B) \cup C \cup A_{dummy} \subseteq N \setminus U$. Moreover, all individuals in $A(B) \cup C \cup A_{dummy}$ must be eliminated in the profile restricted to $N \setminus U$, i.e., $(A(B) \cup C \cup A_{dummy}) \cap f^{(s,2)}(\varphi, N \setminus U) = \emptyset$. Let $a(u)$ be a vertex in $A(B)$ where $u \in B$. Since $\varphi(a(u), a(u)) = 0$, to eliminate $a(u)$, at least one individual that disqualifies $a(u)$ must be in $N \setminus U$. Due to the construction of the profile, all individuals in $N \setminus U$ that disqualify $a(u)$, except $a(u)$ herself, are in $\bigcup_{1 \leq i \leq k} A_i$. Hence, at least one $a_i(v) \in A_i$ where $v \in R_i$ that disqualifies $a(u)$ must be in $N \setminus U$. According to the construction, the vertex v dominates u in the graph G . This implies that $W = \{v \in R \mid a_i(v) \in N \setminus U\}$ dominates B . Now, we show that W contains at most one vertex in each R_i where $1 \leq i \leq k$. Let c_i be an individual in C where $1 \leq i \leq k$. Since $\varphi(c_i, c_i) = 1$, in order to eliminate each c_i , at most $s - 1$ individuals that qualify c_i can be in $N \setminus U$. According to the construction of the profile, all the $s - 3$ dummy individuals in A_{dummy} qualify c_i . Moreover, all individuals in A_i qualify c_i . According to the above discussion, at most one of the individuals in A_i can be in $N \setminus U$. Due to the definition of φ , this implies that $|W \cap R_i| \leq 1$. Now, it is easy to see that W is a solution of the instance \mathcal{I} . \square

4 Conclusion

We have studied the complexity of GCAI, GCDI and GCPI for several well-studied social rules, including the liberal rule, the consent rules and the CSR and LSR rules, where in each problem an external agent has an incentive to make a given subset of individuals socially qualified by adding/deleting/partition of individuals. In particular, we achieved dichotomy results for all three group control problems for consent rules, with respect to the values of the consent quotas. See Table 1 for a summary of these results. In addition, we studied the NP-hard problems from the parameterized complexity point of view, with respect to the size of S , the set of individuals whom the external agent wants to make socially qualified. We proved that GCAI and GCDI for consent rules are generally FPT. For GCPI, however, we proved it remains NP-hard for some consent rules even when $|S| = 1$, excluding the possibility that GCPI for consent is FPT, unless the parameter hierarchy collapse at some level.

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References

- [1] H. Aziz, S. Gaspers, J. Gudmundsson, S. Mackenzie, N. Mattei, and T. Walsh. Computational aspects of multi-winner approval voting. In *AAMAS*, pages 107–115, 2015.
- [2] J. Bang-Jensen and G. Gutin. *Digraphs – Theory, Algorithms and Applications*. Springer-Verlag, London, 2008. 2nd Edition.
- [3] D. Baumeister, G. Erdélyi, E. Hemaspaandra, L. A. Hemaspaandra, and J. Rothe. *Computational Aspects of Approval Voting*, chapter 10, pages 199–251. Handbook on Approval Voting. Springer Berlin Heidelberg, 2010.
- [4] D. B. West. *Introduction to Graph Theory*. Prentice-Hall, 2st, edition, 2000.
- [5] D. Dimitrov. The social choice approach to group identification. In *Consensual Processes*, pages 123–134. 2011.
- [6] D. Dimitrov, S. C. Sung, and Y. Xu. Procedural group identification. *Math. Soc. Sci.*, 54(2):137–146, 2007.
- [7] R. G. Downey and M. R. Fellows. *Parameterized Complexity*. Springer, 1999.

- [8] R. G. Downey and M. R. Fellows. *Fundamentals of Parameterized Complexity*. Texts in Computer Science. Springer, 2013.
- [9] P. Faliszewski, E. Hemaspaandra, and L. A. Hemaspaandra. Using complexity to protect elections. *Commun. ACM.*, 53(11):74–82, 2010.
- [10] P. Faliszewski and A. D. Procaccia. AI’s war on manipulation: Are we winning? *AI. MAG.*, 31(4):53–64, 2010.
- [11] P. C. Fishburn and S. J. Brams. Approval voting, Condorcet’s principle, and runoff elections. *Public. Choice.*, 36(1):89–114, 1981.
- [12] A. Frank and É. Tardos. An application of simultaneous diophantine approximation in combinatorial optimization. *Combinatorica.*, 7(1):49–65, 1987.
- [13] M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, New York, 1979.
- [14] T. F. Gonzalez. Clustering to minimize the maximum intercluster distance. *Theor. Comput. Sci.*, 38:293–306, 1985.
- [15] J. J. Bartholdi III, C. A. Tovey, and M. A. Trick. How hard is it to control an election? *Math. Comput. Model.*, 16(8–9):27–40, 1992.
- [16] R. Kannan. Minkowski’s convex body theory and integer programming. *Math. Oper. Res.*, 12:415–440, 1987.
- [17] D. M. Kilgour and E. Marshall. Approval balloting for fixed-size committees. In D. S. Felsenthal and M. Machover, editors, *Electoral Systems*, Studies in Choice and Welfare, pages 305–326. Springer Berlin Heidelberg, 2012.
- [18] H. W. Lenstra. Integer programming with a fixed number of variables. *Math. Oper. Res.*, 8(4):538–548, 1983.
- [19] A. P. Lin. The complexity of manipulating k -Approval elections. In *ICAART (2)*, pages 212–218, 2011. <http://arxiv.org/abs/1005.4159>.
- [20] A. D. Miller. Group identification. *Game. Econ. Behav.*, 63(1):188–202, 2008.
- [21] H. Nicolas. “I want to be a J!”: Liberalism in group identification problems. *Math. Soc. Sci.*, 54(1):59–70, 2007.
- [22] R. Niedermeier. *Invitation to Fixed-parameter Algorithms*. Oxford University Press Inc, 2006.
- [23] D. Samet and D. Schmeidler. Between liberalism and democracy. *J. Economic Theory*, 110(2):213–233, 2003.
- [24] Y. Yang and J. Guo. The control complexity of r -Approval: from the single-peaked case to the general case. In *AAMAS*, pages 621–628, 2014.
- [25] Y. Yang and J. Guo. Controlling elections with bounded single-peaked width. In *AAMAS*, pages 629–636, 2014.
- [26] Y. Yang and J. Guo. How hard is control in multi-peaked elections: A parameterized study. In *AAMAS*, pages 1729–1730, 2015.

Appendix

The appendix is devoted to the proof of Lemma 1. In particular, we prove the NP-hardness of the LRBDS problem by a reduction from the RED-BLUE DOMINATING SET problem which is NP-hard [13].

RED-BLUE DOMINATING SET (RBDS)

Input: A bipartite graph $B = (R \uplus B, E)$ and an integer k .

Question: Is there a subset $W \subseteq R$ such that $|W| \leq k$ and W dominates B ?

Let $I' = (G' = (R' \uplus B', E'), k)$ be an instance of the RBDS problem. We construct an instance $I = (G = (R \uplus B, E), \{1, 2, \dots, k\})$ for the LRBDS problem as follows. For each vertex $u \in B'$, we create a vertex $\bar{u} \in B$. For each vertex $v \in R'$, we create k vertices $v(1), \dots, v(k) \in R$, where the vertex $v(i)$ is labeled with i . Let R_i be the set of the vertices in R that have label i . The edges of the graph G are defined as follows. If there is an edge $\{v, u\} \in E'$, then for every $1 \leq i \leq k$ we create an edge between $v(i)$ and \bar{u} . This finishes the construction. It clearly takes polynomial time.

Suppose that I' has a solution W' of size $k' \leq k$. Let $(v_{x(1)}, v_{x(2)}, \dots, v_{x(k')})$ be any arbitrary order of the vertices in W' . Let $W = \{v(i) \mid v_{x(i)} \in W', 1 \leq i \leq k'\}$. It is clear that no two vertices in W have the same label, that is, $|W \cap R_i| \leq 1$ for every $1 \leq i \leq k$. We shall show that W dominates B . Let u be a vertex in B' . Since W' dominates B' , there is a vertex $v_{x(i)} \in W'$ such that $\{v_{x(i)}, u\} \in E'$. Then, according to the construction of G , we know that $\{v(i), \bar{u}\} \in E$. Since this holds for every $u \in B'$, W dominates B .

Suppose that I has a solution W . We assume that for every $v \in R'$, W contains at most one of $\{v(1), v(2), \dots, v(k)\}$. Indeed if W contains two vertices $v(i)$ and $v(j)$ where $1 \leq i \neq j \leq k$, then we could get a new solution $W \setminus \{v(j)\}$ for I , since according to the construction of the graph G , $v(i)$ and $v(j)$ have the same neighborhood, implying that a vertex in B dominated by $v(j)$ is dominated by $v(i)$. Let $W' = \{v \in R' \mid v(i) \in W, 1 \leq i \leq k\}$. Let u be a vertex in B' . Since W is a solution of I , there is a vertex $v(i) \in W$ which dominates $\bar{u} \in B$. Then, according to the construction of the graph G , the vertex $v \in W'$ dominates u . It follows that W' dominates B' .